

Evolutionary quantum cosmology in a gauge-fixed picture

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Abstract

We study the classical and quantum models of a flat Friedmann-Robertson-Walker (FRW) space-time, coupled to a perfect fluid, in the context of the consensus and a gauge-fixed Lagrangian frameworks. It is shown that, either in the usual or in the gauge-fixed actions, the evolution of the Universe based on the classical cosmology represents a late time power law expansion, coming from a big-bang singularity in which the scale factor goes to zero for the standard matter, and tending towards a big-rip singularity in which the scale factor diverges for the phantom fluid. We then employ the familiar canonical quantization procedure in the given cosmological setting to find the cosmological wave functions in the corresponding minisuperspace. Using a gauge-fixed (reduced) Lagrangian, we show that, it may lead to a Schrödinger equation for the quantum-mechanical description of the model under consideration, the eigenfunctions of which can be used to construct the time dependent wave function of the Universe. We use the resulting wave function in order to investigate the possibility of the avoidance of classical singularities due to quantum effects by means of the many-worlds and ontological interpretation of quantum cosmology.

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1 Introduction

The standard model of relativistic cosmology has its origin in the general theory of relativity. As is well known, standard cosmological models based on classical general relativity have no convincingly precise answer to the question of the initial conditions from which the Universe has evolved. This can be traced to the fact that these models suffer from the presence of an initial singularity, the so-called big-bang singularity. Indeed, there are various forms of singularity theorems in general relativity [1] which show that quite reasonable assumptions lead to at least one consequence which is physically unacceptable. Any hope of dealing with such singularities would be in the development of a concomitant and conducive quantum theory of gravity [2]. In the absence of a full theory of quantum gravity, it would be useful to describe the quantum states of the Universe within the context of quantum cosmology. In this formalism which is based on the canonical quantization procedure, one first freezes a large number of degrees of freedom and then quantizes the remaining ones. The quantum state of the Universe is then described by a wave function in the minisuperspace, a function of the 3-geometry and matter fields presented in the theory, satisfying the Wheeler-DeWitt equation [3]. An important feature of this wave function is that unlike the case of ordinary quantum mechanics, the wave function in quantum cosmology is time independent. This is indeed a reflect of the fact that general relativity is a parameterized theory in the sense that its (Einstein-Hilbert) action is invariant under time reparameterization. Because the existence of the "problem of time", that is, time evolution is lost in the dynamics of the wave function, the canonical formulation of general

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relativity leads to a constrained system and its Hamiltonian is a superposition of some constraints, the so-called Hamiltonian and momentum constraints. A possible way to overcome this problem is that one first solves the equation of constraint to obtain a set of genuine canonical variables with which one can construct a reduced Hamiltonian. In this kind of time reparameterization, the equations of motion are obtained from the reduced physical Hamiltonian and describe the evolution of the system with respect to the selected time parameter [4].

An important ingredient in any model theory related to the quantization of a cosmological model is the choice of the quantization procedure used to quantize the system. As mentioned above, the most widely used method has traditionally been the canonical quantization method based on the Wheeler-DeWitt equation which is nothing but the application of the Hamiltonian constraint to the wave function of the Universe. However, one may solve the constraint before using it in the theory and, in particular, before quantizing the system. Then, one can quantize the reduced system in the same manner as in elementary quantum mechanics. If we do so, since there are no constraints in the reduced phase space, we are led to a Schrödinger type equation where a time reparameterization in terms of various dynamical variables can be done before quantization [5]. Although such kinds of gauge fixing lead to different time parameters, the physical behavior of the system under consideration remains invariant under the corresponding gauge transformations. There is also another type of gauge fixing proposed in [6], and this is removing the gauge freedom from the theory at the level of Lagrangian. In this method, instead of the usual Einstein-Hilbert Lagrangian one can construct a Lagrangian theory for the gravitational system free of the traditional gauge freedoms and show that the correct dynamics of the system is recovered while the equations of motion are different.

In this paper, we first deal with a spatially flat FRW cosmology with a perfect fluid as its matter field. Classically, we show that the corresponding dynamical system resulting from the Einstein-Hilbert action has a compatible set of solutions and constraints. Quantum mechanically, we employ the familiar canonical quantization procedure in this cosmological setting and investigate the behavior of the resulting wave function either in the semiclassical and quantum regimes. We then consider the problem at hand in the context of a gauge-fixed Lagrangian proposed in [6]. We see that the examined action is free of the usual Hamiltonian constraint and thus it may lead to the identification of a time parameter for the corresponding dynamical system. Moreover, this formalism gives rise to a Schrödinger equation for the quantum-mechanical description of the model under consideration. Finally, we use the resulting wave function in order to investigate the possibility of the avoidance of classical singularities due to quantum effects by means of the many-worlds and ontological interpretation of quantum cosmology.

2 Perfect fluid cosmology

In this section, we start by briefly studying the ordinary, spatially flat FRW model, where the metric is given by

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (1)$$

with $N(t)$ and $a(t)$ being the lapse function and the scale factor, respectively. Also, we assume that a perfect fluid with which the action of the theory is augmented plays the role of the matter part of the model. The Einstein-Hilbert Lagrangian with a general energy density $V(a)$ becomes [7]

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}(R[g] - V(a)) \\ &= -6N^{-1}a\dot{a}^2 - Na^3V(a), \end{aligned} \quad (2)$$

where $R[g]$ is the Ricci scalar and in the second line the total derivative term has been ignored. For a perfect fluid where its pressure p is linked to its energy density ρ by a barotropic Equation of State (EoS)

$$p = \omega\rho, \quad (3)$$

the function $V(a)$ becomes $V(a) = M_\omega a^{-3(\omega+1)}$. Here, M_ω is a model dependent constant and $-1 \leq \omega \leq 1$ is the EoS parameter. Therefore, by considering the above action as representing a dynamical system in which the scale factor a plays the role of the dynamical variable, we obtain the following pointlike Lagrangian

$$\mathcal{L} = -6N^{-1}a\dot{a}^2 - NM_\omega a^{-3\omega}. \quad (4)$$

2.1 The classical model

The classical dynamics are governed by the Euler-Lagrange equation $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} - \frac{\partial \mathcal{L}}{\partial a} = 0$, that is,

$$4a\ddot{a} + 2\dot{a}^2 + N\omega M_\omega a^{-3\omega-1} = 0. \quad (5)$$

As is well known the cosmological model, in view of the concerning issue of time, has been rather general and of course under-determined. Before trying to solve this equation we must decide on a choice of time in the theory. The under determinacy problem at the classical level may be removed by using the gauge freedom via fixing the gauge. For example, we can work in the gauge $N = 1$, which usually is chosen in classical cosmological models and is called the cosmic time gauge. To proceed, we consider the gauge $N = 1$ and assume that the scale factor depends on the cosmic time as

$$a(t) = (At - \tau_0)^\beta, \quad (6)$$

where A , τ_0 , and β are some constants. Using this ansatz in Eq. (5), it is straightforward to obtain

$$2\beta - 2 = -(3\omega + 1)\beta, \quad 2\beta(2 - 3\beta)A^2 - \omega M_\omega = 0, \quad (7)$$

where their solutions (for $\omega \neq -1$) can be written as

$$\beta = \frac{2}{3(\omega + 1)}, \quad A = \sqrt{\frac{3M_\omega}{8}}(\omega + 1). \quad (8)$$

Therefore, the classical cosmology exhibits a power law expansion with the scale factor

$$a(t) = \left[\sqrt{\frac{3M_\omega}{8}}(\omega + 1)t - \tau_0 \right]^{\frac{2}{3(\omega+1)}}. \quad (9)$$

For $\omega = -1$, the perfect fluid plays the role of a cosmological constant and in this case we can directly integrate the Eq. (5) to obtain

$$a(t) = a_0 \exp \left(\sqrt{\frac{M_{-1}}{6}} t \right). \quad (10)$$

As a double check, one may obtain the above solutions from the Einstein equations. Indeed, assuming the full Einstein equations to hold, this implies that the Hamiltonian corresponding to the Lagrangian (4) must vanish, that is

$$-6a\dot{a}^2 + M_\omega a^{-3\omega} = 0. \quad (11)$$

It is easy to check that the solutions (9) and (10) automatically satisfy the Hamiltonian constraint (11), and thus we have a compatible system of solutions and constraint. The evolution of the Universe based on (9) begins with a big-bang singularity at $t = \sqrt{\frac{8}{3M_\omega}} \frac{\tau_0}{\omega+1}$ and, for $\omega > -1$, follows the power law expansion $a(t) \sim t^{\frac{2}{3(\omega+1)}}$ at the late time of cosmic evolution. We can also compute the deceleration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2}$ for this model. As is well known the deceleration parameter indicates by how much the expansion of the Universe is slowing down. If the expansion is speeding up, for which there appears to be some recent evidence, then this parameter will be negative. A simple calculation shows that at late time limit we have $q \sim \frac{3\omega+1}{2}$, and therefore for the EoS parameter in the range $-1 < \omega < -\frac{1}{3}$ the late time expansion will occur with a positive acceleration.

In the case of a phantom fluid $\omega < -1$, the solution corresponding to the expanding scale factor reads as [8]

$$a_{ph}(t) \sim (t_{rip} - t)^{-\frac{2}{3|\omega+1|}}. \quad (12)$$

In this case t is smaller than the constant t_{rip} , and thus as $t \rightarrow t_{rip}$, the scale factor diverges. The evolution of the Universe ends up with a finite-time singularity, the so-called big-rip singularity [9, 10]. Since the phantom energy density is proportional to the scale factor as $\rho \sim a^{3|\omega+1|}$, at the big-rip the energy density and pressure also diverge. One can easily see that this is different from the ordinary big-crunch singularity at which the energy density and pressure blow up as the scale factor tends to zero at a finite time. In Refs. [10] some exact solutions of classical phantom cosmology were studied. These solutions show that including a cosmological constant into the model gives rise to a scale factor which its evolution, for standard types of matter, begins with big-bang and terminates at big-crunch, while for the phantom case, it begins with big-rip and terminates also at a big-rip. An interesting remark of the phantom solution (12) is that it looks to be dual of the standard matter solution (9) under the duality transformation

$$a \leftrightarrow \frac{1}{a}, \quad \omega + 1 \leftrightarrow |\omega + 1|. \quad (13)$$

Such duality is one of the major features of the solutions of equations of motion in phantom cosmology, so that if the set $(a, \omega + 1)$ solves the equations of motion for the standard matter models, the set $(a^{-1}, |\omega + 1|)$ solves the same equations for the phantom model. In the following we shall see that how these classical pictures will be modified if one takes into account the quantum-mechanical considerations in the problem at hand.

2.2 The quantum model

We now focus attention on the study of the quantum cosmology of the models described above. The momentum conjugate to a is

$$P_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -12N^{-1}a\dot{a}, \quad (14)$$

giving rise to the following Hamiltonian

$$H = N\mathcal{H} = N \left[-\frac{P_a^2}{24a} + M_\omega a^{-3\omega} \right]. \quad (15)$$

We now quantize the system with the use of the Wheeler-DeWitt equation, that is, $\mathcal{H}\Psi = 0$, where \mathcal{H} is the operator form of the Hamiltonian given by the above equation, and Ψ is the wave function of the Universe, a function of the scale factor and the matter fields, if they exist. With the replacement $P_a \rightarrow -i\frac{d}{da}$ we get the Wheeler-DeWitt equation as

$$\left(\frac{d^2}{da^2} + \frac{p}{a} \frac{d}{da} + 24M_\omega a^{1-3\omega} \right) \Psi(a) = 0, \quad (16)$$

where the parameter p represents the ambiguity in the ordering of factors a and P_a in the first term of (15). For large values of a , the solution of the above equation can be easily obtained in the WKB (semiclassical) approximation. In this regime we can neglect the second term in Eq. (16). Then substituting $\Psi(a) = \Omega(a)e^{iS(a)}$ in this equation leads to the modified Hamilton-Jacobi equation

$$-\left(\frac{dS}{da} \right)^2 + 24M_\omega a^{1-3\omega} + \mathcal{Q} = 0, \quad (17)$$

in which the quantum potential is defined as $\mathcal{Q} = \frac{1}{\Omega} \frac{d^2 \Omega}{da^2}$. It is well known that the quantum effects are important for small values of the scale factor and in the limit of the large scale factor can be

neglected. Therefore, in the semiclassical approximation region we can omit the \mathcal{Q} term in Eq. (17) and obtain the phase function $S(a)$ as

$$S = \pm \frac{2\sqrt{24M_\omega}}{3(1-\omega)} a^{\frac{3(1-\omega)}{2}}, \quad (18)$$

where the positive sign corresponds to an expanding Universe. In the WKB method, the correlation between classical and quantum solutions is given by the relation $P_a = \frac{\partial S}{\partial a}$. Thus, using the definition of P_a in (14), the equation for the classical trajectories becomes

$$-12a\dot{a} = \sqrt{24M_\omega} a^{\frac{1-3\omega}{2}}, \quad (19)$$

from which one finds

$$a(t) \sim \left(\sqrt{\frac{3M_\omega}{8}} (1+\omega)t \right)^{\frac{2}{3(1+\omega)}}, \quad (20)$$

where shows that the late time behavior of the classical cosmology (9) is exactly recovered. The meaning of this result is that for large values of the scale factor the effective action corresponding to the expanding Universe is very large and the Universe can be described classically. On the other hand, for small values of the scale factor we cannot neglect the quantum effects and the classical description breaks down. Since the WKB approximation is no longer valid in this regime, one should go beyond the semiclassical approximation. In general, the two linearly independent solutions to Eq. (16) can be expressed in terms of the Bessel functions J and Y leading to the following general solution

$$\Psi(a) = a^{\frac{1-p}{2}} \left[c_1 J_{\frac{1-p}{3(1-\omega)}} \left(\frac{4\sqrt{6M_\omega}}{3(1-\omega)} a^{\frac{3(1-\omega)}{2}} \right) + c_2 Y_{\frac{1-p}{3(1-\omega)}} \left(\frac{4\sqrt{6M_\omega}}{3(1-\omega)} a^{\frac{3(1-\omega)}{2}} \right) \right]. \quad (21)$$

Note that the minisuperspace of the above model has only one degree of freedom denoted by the scale factor a in the range $0 < a < +\infty$. According to [11], its nonsingular boundary is the line $a = 0$, while at the singular boundary this variable is infinite. Now, we impose the boundary condition on the above solutions such that at the nonsingular boundary the wave function vanishes [11], which yields $c_2 = 0$, and thus we arrive at the unique solution

$$\Psi(a) = a^{\frac{1-p}{2}} J_{\frac{1-p}{3(1-\omega)}} \left(\frac{4\sqrt{6M_\omega}}{3(1-\omega)} a^{\frac{3(1-\omega)}{2}} \right). \quad (22)$$

Note that Eq. (16) is a Schrödinger-like equation for a fictitious particle with zero energy moving in the field of the superpotential $U(a) = -24M_\omega a^{1-3\omega}$. Usually, in the presence of such a potential the minisuperspace can be divided into two regions, $U > 0$ and $U < 0$, which could be termed the classically forbidden and classically allowed regions respectively. In the classically forbidden region the behavior of the wave function is exponential while in the classically allowed region the wave function behaves oscillatory. In the quantum tunneling approach [11], the wave function is so constructed as to create a Universe emerging from *nothing* by a tunneling procedure through a potential barrier in the sense of usual quantum mechanics. Now, in our model the superpotential is always negative which means that there is no possibility of tunneling anymore since a zero energy system is always above the superpotential. In such a case tunneling is no longer required as classical evolution is possible. As a consequence the wave function always exhibits oscillatory behavior. In figure 1 we have plotted the square of the wave functions for typical values of the parameters. It is seen from this figure that the wave function has a well-defined behavior near $a = 0$ and describes a Universe emerging out of *nothing* without any tunneling. (See [12] in which such a wave functions are also appeared in the case study of the probability of quantum creation of compact flat and open de Sitter (dS) Universes.) On the other hand, the emergence of several peaks in the wave function may be interpreted as a representation of different quantum states that may communicate with each other through tunneling. This means that there are different possible Universes (states) from which our present Universe could have evolved and tunneled in the past, from one Universe (state) to another.

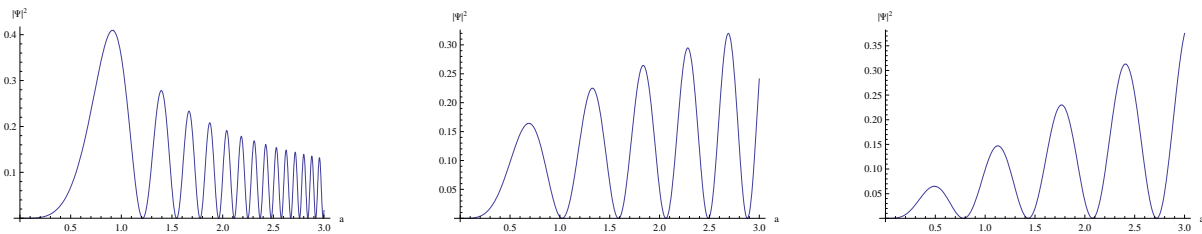


Figure 1: The square of the wave function for the quantum Universe. We take the numerical values $M_\omega = 1$, $p = -1$ and $\omega = -1, 0, 1/3$.

3 Gauge-fixed perfect fluid cosmology

In this section we deal with the problem at hand in an another point of view. As is well known from the classical mechanics, the Lagrangian of a given dynamical system is unique up to a total derivative of time. This means that if L is a Lagrangian for a dynamical system satisfying Lagrange's equations then $L' = L + \frac{df}{dt}$ also satisfies Lagrange's equations, where f is a differentiable function of coordinates and time. Therefore, with different choices of the function f we obtain different equivalent Lagrangians which are called trivial or gauge equivalent Lagrangians. On the other hand, for such a dynamical system one can demonstrate that very different Lagrangians lead to the same equation of motion. For instance, for a one dimensional particle moving under the act of potential $V(x)$, in addition of the usual Lagrangian $L = \frac{1}{2}m\dot{x}^2 - V(x)$ and its gauge equivalent, one can show that the Lagrangian $L = \frac{1}{12}m^2\dot{x}^4 + m\dot{x}^2V(x) - V^2(x)$ also gives the correct equation of motion [13]. The same is true for the Lagrangians $L = \frac{1}{n}\dot{x}^n$ ($n \geq 2$) for a one dimensional free particle and $L = T - V + \frac{\gamma J}{r^2}$ (γ is a constant and J is the angular momentum) for a particle moving under a spherically symmetric potential. The problem of finding these so-called nontrivial equivalent Lagrangians is the subject of the inverse problem in the calculus of variation [14]. Although from the (nontrivial) equivalent Lagrangians we obtain the same classical equations of motion, canonical quantization of the system under consideration based on these Lagrangians may yield different results. Hence, classical equivalent systems may be nonequivalent quantum mechanically. In nonlinear systems (like gravitational field equations) one can still think about another kinds of Lagrangians which give rise to the correct dynamics for the system while the equations of motion may be different with the ones coming from the usual Lagrangian. In the next subsection we shall introduce such a Lagrangian for our model at hand and see that although its equation of motion is somehow different with once of Einstein-Hilbert action, the solutions are the same. In this sense we refer these kinds of Lagrangians also as the non trivial equivalent with the traditional Einstein-Hilbert Lagrangian.

3.1 The classical model

Now, we consider again the flat FRW Universe (1) filled with a perfect fluid with EoS (3). Instead of the consensus Lagrangian (4) we examine the following (up to a overall factor) Lagrangian proposed in [6]

$$\mathcal{L} = a^{3\omega+1}\dot{a}^2. \quad (23)$$

A simple calculation based on the Euler-Lagrange equation gives the equation of motion as

$$(3\omega + 1)\dot{a}^2 + 2a\ddot{a} = 0, \quad (24)$$

where its solution reads

$$a(t) = \mathcal{A} [3(\omega + 1)t - \mathcal{B}]^{\frac{2}{3(\omega+1)}}, \quad (25)$$

for $\omega \neq -1$, and

$$a(t) = \mathcal{C}_1 e^{\mathcal{C}_2 t}, \quad (26)$$

for $\omega = -1$. We may set the integration constants \mathcal{A} , \mathcal{B} , \mathcal{C}_1 and \mathcal{C}_2 as $\mathcal{A} = \left(\frac{M_\omega}{24}\right)^{\frac{1}{3(\omega+1)}}$, $\mathcal{B} = \tau_0 \left(\frac{24}{M_\omega}\right)^{1/2}$, $\mathcal{C}_1 = a_0$ and $\mathcal{C}_2 = \sqrt{\frac{M_\omega-1}{6}}$ so that the above solutions coincide with the expressions (9) and (10) for the corresponding classical cosmology. Therefore, although the difference between the Lagrangians (23) and (4) is not a total derivative of time and these two Lagrangians yield different classical equations of motion (5) and (24), their equations of motion have the same solutions. As mentioned above, we refer the Lagrangian (23) as an (nontrivial) equivalent Lagrangian for the system under consideration. Now, the question is that what the relation between two Lagrangians is? As is well known, general relativity is a parameterized gauge theory which its canonical formalism is based on the Hamiltonian and momentum constraints. Because of the existence of these constraints the quantum version of this theory suffers from a number of problems, namely the construction of the Hilbert space to define a positive definite inner product of the solutions of the Wheeler-DeWitt equation, the operator ordering problem, and also, most importantly, the problem of time. As we have seen in the previous section, the wave function in the Wheeler-DeWitt equation is independent of time, i.e., the Universe has a static picture in this scenario. This problem was first addressed in [2] by DeWitt himself. However, he argued that the problem of time should not be considered as a hindrance in the sense that the theory itself must include a suitable well-defined time in terms of its geometry or matter fields. In this scheme time is identified with one of the characters of the geometry or with a scalar character of matter fields coupled to gravity in any specific model. Identification of time with one of the dynamical variables depends on the method we use to deal with the constraints. In any constrained system we can impose the constraints in different steps. In classical mechanics, for example, we may first solve the equations of constraint to reduce the degrees of freedom of the system and obtain a minimal number of dynamical variables. On the other hand, we may multiply the constraint by a variable parameter and add it to the Lagrangian. This Lagrange multiplier plays the role of an additional dynamical variable and the equations of motion consist of those obtained from variation of the Lagrangian with respect to the dynamical variables plus the equation of constraint. Also, when quantizing the system, we may impose the constraint before or after the quantization has been done. Now, if our system is the entire Universe, e.g., in the case of quantum cosmology, these procedures result in different approaches to the problem of time reparameterization [5].

In writing Lagrangian (23), as is argued in [6] the gauge freedom is removed at the level of Lagrangian and the system is reduced to its true degree of freedom. Hence, the Hamiltonian of the model is not expected to vanish identically. Therefore, in quantization of such a system we should deal with a Schrödinger equation instead of the Wheeler-DeWitt equation in which we have a time dependent wave function.

Let us now set up the Hamiltonian formalism of the theory based on the Lagrangian (23). We introduce the conjugate momentum as

$$P_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = 2a^{3\omega+1} \dot{a}, \quad (27)$$

and by the usual Legendre transformation we obtain the Hamiltonian as

$$\mathcal{H} = \frac{P_a^2}{4a^{3\omega+1}}. \quad (28)$$

It is easy to see that the classical Hamiltonian equations admit the solution (25). Now, if using (25), we compute the numerical value of the Hamiltonian (28) we get

$$\mathcal{H} = 4\mathcal{A}^{3(\omega+1)} = E = \text{const.} \quad (29)$$

As we have mentioned above, in contrast to the Hamiltonian constraint in parameterized theory in the previous section, in the gauge-fixed theory based on the equivalent Lagrangian the corresponding Hamiltonian is not identically equal to zero.

3.2 The quantum model

To pass to the quantum version of this model we should note that the usual approach to canonical quantization of a cosmological model is the Wheeler-DeWitt approach where one uses the Dirac method to quantize the degrees of freedom of the system. The role of constraints in their operator form is to annihilate the wave function of the Universe. This procedure leads one to the basic equation of quantum cosmology, the so-called Wheeler-DeWitt equation. This is what we have performed in the previous section. However, as was done in this section, one may fix the constraint before using it in the theory, in particular before quantizing the system. If we do so, we are led to the Schrödinger equation

$$\mathcal{H}\Psi(a, t) = i\frac{\partial}{\partial t}\Psi(a, t), \quad (30)$$

where \mathcal{H} is the operator form of the reduced Hamiltonian (28) and $\Psi(a, t)$ is the time dependent wave function of the Universe. With the usual replacement $P_a \rightarrow -i\partial_a$ and choice of a particular factor ordering this equation becomes

$$\frac{1}{4} \left(\frac{3\omega + 1}{2} a^{-3\omega-2} \frac{\partial}{\partial a} - a^{-3\omega-1} \frac{\partial^2}{\partial a^2} \right) \Psi(a, t) = i\frac{\partial}{\partial t}\Psi(a, t). \quad (31)$$

We separate the variables in the Schrödinger Eq. (31) as

$$\Psi(a, t) = e^{-iEt}\psi(a), \quad (32)$$

leading to

$$\left(\frac{3\omega + 1}{2} a^{-3\omega-2} \frac{d}{da} - a^{-3\omega-1} \frac{d^2}{da^2} - 4E \right) \psi(a) = 0. \quad (33)$$

For $\omega \neq -1$ the solutions of the above differential equation may be written in the form

$$\psi(a) = d_1 \sin \left[\frac{4\sqrt{E}}{3(\omega + 1)} a^{\frac{3(\omega+1)}{2}} \right] + d_2 \cos \left[\frac{4\sqrt{E}}{3(\omega + 1)} a^{\frac{3(\omega+1)}{2}} \right], \quad (34)$$

where d_1 and d_2 are integration constants. If we impose the boundary condition $\psi(a = 0) = 0$ on these solutions, the eigenfunctions of the Schrödinger equation can be written as

$$\Psi_E(a, t) = e^{-iEt} \sin \left[\frac{4\sqrt{E}}{3(\omega + 1)} a^{\frac{3(\omega+1)}{2}} \right]. \quad (35)$$

We may now write the general solution to the Schrödinger equation as a superposition of its eigenfunctions, that is

$$\Psi(a, t) = \int_0^\infty A(E) \Psi_E(a, t) dE, \quad (36)$$

where $A(E)$ is a suitable weight function to construct the wave packets. By using the equality [15]

$$\int_0^\infty e^{-\gamma x} \sin \sqrt{mx} dx = \frac{\sqrt{\pi m}}{2\gamma^{3/2}} e^{-\frac{m}{4\gamma}}, \quad (37)$$

we can evaluate the integral over E in (36) and simple analytical expression for this integral is found if we choose the function $A(E)$ to be a quasi-Gaussian weight factor $A(E) = e^{-\gamma E}$ (γ is an arbitrary positive constant), which results in

$$\Psi(a, t) = \int_0^\infty e^{-\gamma E} e^{-iEt} \sin \left[\frac{4\sqrt{E}}{3(\omega + 1)} a^{\frac{3(\omega+1)}{2}} \right] dE. \quad (38)$$

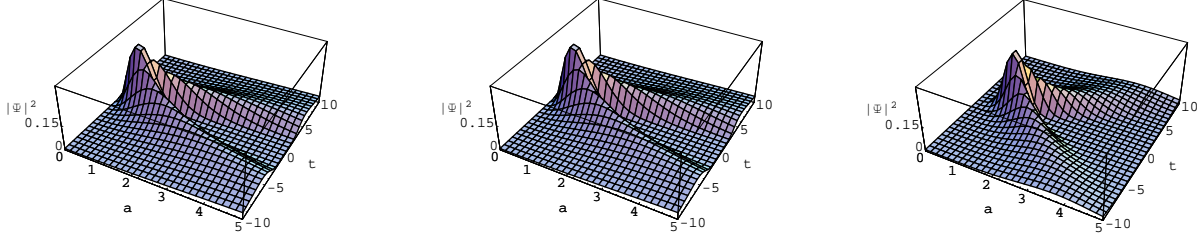


Figure 2: The figures show $|\Psi(a, t)|^2$, the square of the wave function. The figures are plotted for the numerical values $\mathcal{N} = \gamma = 1$, and we set the equation of state parameter (from left to right) $\omega = -1/3, 0, 1/3$. After examining some other values for this parameter, we verify that the general behavior of the wave function is repeated.

Using of the relation (37) leads to the following expression for the wave function

$$\Psi(a, t) = \mathcal{N} \frac{a^{\frac{3(\omega+1)}{2}}}{(\omega+1)(\gamma+it)^{3/2}} \exp \left[-\frac{4a^{3(\omega+1)}}{9(\omega+1)^2(\gamma+it)} \right], \quad (39)$$

where \mathcal{N} is a numerical factor. Now, having the above expression for the wave function of the Universe, we are going to obtain the predictions for the behavior of the dynamical variables in the corresponding cosmological model. To do this, in the next subsection, we shall adopt two approaches to evaluate the classical behavior of the dynamical variables in the model which lead to the same results. In the many-worlds interpretation of quantum mechanics [16], we can calculate the expectation values of the dynamical variables and, in the realm of the ontological interpretation of quantum mechanics [17], one can evaluate the Bohmian trajectories for those variables. In figure 2 we have plotted the square of the wave function for typical numerical values of the parameters. As this figure shows, at $t = 0$, the wave function has a dominant peak in the vicinity of some nonzero value of a . This means that the wave function predicts the emergence of the Universe from a nonsingular state corresponding to this dominant peak. As time progresses, the wave packet begins to propagate in the a direction, its width becoming wider and its peak moving towards the greater values of a . The wave packet disperses as time passes, the minimum width being attained at $t = 0$. As in the case of the free particle in quantum mechanics, the more localized the initial state at $t = 0$, the more rapidly the wave packet disperses. Therefore, the quantum effects make themselves felt only for small enough t corresponding to small a as expected, and the wave function predicts that the Universe will assume states with larger a in its late time evolution.

The above solutions are not valid for $\omega = -1$. In this particular case, Eq. (33) becomes an Euler's type equation, and following the steps (34)-(38) the final result for the wave function takes the form

$$\Psi(a, t) = \mathcal{N} \frac{\ln a}{(\gamma+it)^{3/2}} \exp \left[-\frac{\ln^2 a}{\gamma+it} \right]. \quad (40)$$

The behavior of this function is represented in figure 3. The discussions on the comparison between quantum cosmological solution and its classical counterpart are the same as previous models, namely the $\omega \neq -1$ models. Similar discussion as above would be applicable to this case as well.

3.3 Recovery of the classical solutions

In general, one of the most important features in quantum cosmology is the recovery of classical cosmology from the corresponding quantum model or, in other words, how can the quantum wave functions predict a classical Universe. In this approach, one usually constructs a coherent wave packet with good asymptotic behavior in the minisuperspace, peaking in the vicinity of the classical trajectory. On the other hand, in an another approach to show the correlations between classical

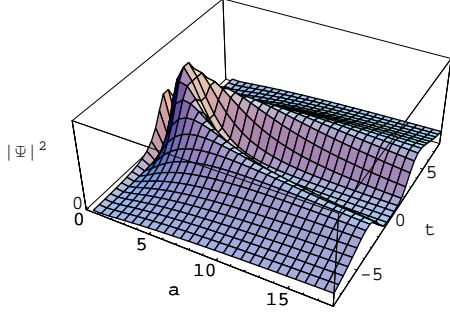


Figure 3: The figure shows $|\Psi(a, t)|^2$, the square of the wave function for $\omega = -1$. The figure is plotted for the numerical values $\mathcal{N} = 1$ and $\gamma = 1.5$.

and quantum pattern, following the many-worlds interpretation of quantum mechanics [16], one may calculate the time dependence of the expectation value of a dynamical variable q as

$$\langle q \rangle (t) = \frac{\langle \Psi | q | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \quad (41)$$

Following this approach, we may write the expectation value for the scale factor as

$$\langle a \rangle (t) = \frac{\int_0^\infty \Psi^*(a, t) a \Psi(a, t) da}{\int_0^\infty \Psi^*(a, t) \Psi(a, t) da}, \quad (42)$$

which yields

$$\langle a \rangle (t) = \left(\frac{9(\omega + 1)^2}{8\gamma} \right)^{\frac{1}{3(\omega+1)}} \frac{\Gamma\left(\frac{5+3\omega}{3(1+\omega)}\right)}{\Gamma\left(\frac{4+3\omega}{3(1+\omega)}\right)} (\gamma^2 + t^2)^{\frac{1}{3(1+\omega)}}. \quad (43)$$

It is important to classify the nature of the quantum model as concerns the presence or absence of singularities. For the wave function (39), the expectation value (43) of a never vanishes, showing that these states are nonsingular. Indeed, in (43) t varies from $-\infty$ to $+\infty$ and $t = 0$ is just a specific moment without any particular physical meaning like big-bang singularity. The expression (43) for $\omega \neq -1$, represents a bouncing Universe with no singularity where its late time behavior coincides to the late time behavior of the classical solution (9), that is $a(t) \sim t^{\frac{2}{3(\omega+1)}}$. Now we can calculate the dispersion of the wave packet in the a direction, which is defined as

$$(\Delta a)^2 = \langle a^2 \rangle - \langle a \rangle^2, \quad (44)$$

using (39) and (43), we get

$$(\Delta a)^2 \sim (\gamma^2 + t^2)^{\frac{2}{3(1+\omega)}}. \quad (45)$$

The result is that the wave packet traveling in the a direction spreads as time increases and thus its degree of localization is reduced. The width of the wave packet evaluated in (45) agrees with the discussion in the end of the previous subsection. Indeed, we may interpret the above relation for the width of the wave function as the coincidence of the classical trajectories with the quantum ones for large values of time. Therefore, in view of the behavior of the scale factor, the classical solution (43) is in complete agreement with the quantum patterns shown in figure 2, and both predict a (nonsingular) monotonically increasing evolution for the scale factor and consequently there is an almost good correlation between the quantum patterns and classical trajectories.

The issue of the correlation between classical and quantum schemes may be addressed from another point of view. It is known that the results obtained by using the many-world interpretation agree

with those that can be obtained using the ontological interpretation of quantum mechanics [17]. In Bohmian interpretation, the wave function is written as

$$\Psi(a, t) = \Omega(a, t)e^{iS(a, t)}, \quad (46)$$

where Ω and S are some real functions. Substitution of this expression into the Schrödinger Eq. (31) leads to the continuity equation

$$\frac{1}{4} \left[\frac{3\omega + 1}{2} a^{-3\omega-2} \Omega \frac{\partial S}{\partial a} - 2a^{-3\omega-1} \frac{\partial S}{\partial a} \frac{\partial \Omega}{\partial a} - a^{-3\omega-1} \frac{\partial^2 S}{\partial a^2} \right] - \frac{\partial \Omega}{\partial t} = 0, \quad (47)$$

and the modified Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{4} a^{-3\omega-1} \left(\frac{\partial S}{\partial a} \right)^2 + \mathcal{Q} = 0, \quad (48)$$

in which the quantum potential \mathcal{Q} is defined as

$$\mathcal{Q} = \frac{1}{4\Omega} \left[\frac{3\omega + 1}{2} a^{-3\omega-2} \frac{\partial \Omega}{\partial a} - a^{-3\omega-1} \frac{\partial^2 \Omega}{\partial a^2} \right]. \quad (49)$$

From the wave function (39), the real functions $\Omega(a, t)$ and $S(a, t)$ can be obtained as

$$\Omega(a, t) = \mathcal{N} \frac{a^{\frac{3(\omega+1)}{2}}}{(\omega+1)(\gamma^2 + t^2)^{3/4}} \exp \left[-\frac{4\gamma}{9(\omega+1)^2(\gamma^2 + t^2)} a^{3(\omega+1)} \right], \quad (50)$$

and

$$S(a, t) = -\frac{3}{2} \arctan \frac{t}{\gamma} + \frac{4t}{9(\omega+1)^2(\gamma^2 + t^2)} a^{3(\omega+1)}. \quad (51)$$

In this interpretation the classical trajectories, which determine the behavior of the scale factor are given by $P_a = \frac{\partial S}{\partial a}$. Using the expression for P_a in (27), the equation for the classical trajectories becomes

$$2a^{3\omega+1} \dot{a} = \frac{4t}{3(\omega+1)(\gamma^2 + t^2)} a^{3\omega+2}. \quad (52)$$

Therefore, after integration we get

$$a(t) = a_0 \left(\gamma^2 + t^2 \right)^{\frac{1}{3(\omega+1)}}, \quad (53)$$

where a_0 is a constant of integration. This solution has the same behavior as the expectation value computed in (43) and like that is free of singularity. The origin of the singularity avoidance may be understood by the existence of the quantum potential which corrects the classical equations of motion. Inserting the relation (53) in (50) and then using (49), we can find the quantum potential in terms of the scale factor as

$$\mathcal{Q} \sim a^{-3(\omega+1)}. \quad (54)$$

It is obvious from this equation that the quantum effects are important for small values of the scale factor and in the limit of the large scale factor can be neglected. Therefore, asymptotically the classical behavior is recovered. In this sense we can extract a repulsive force from the quantum potential (54) as

$$F_a = -\frac{\partial \mathcal{Q}}{\partial a} \sim a^{-(3\omega+4)}, \quad (55)$$

which may interpreted as being responsible of the avoidance of singularity. For small values of a (near the big-bang singularity), this repulsive force takes a large magnitude and thus prevents the scale factor to evolve to the classical singularity $a \rightarrow 0$.

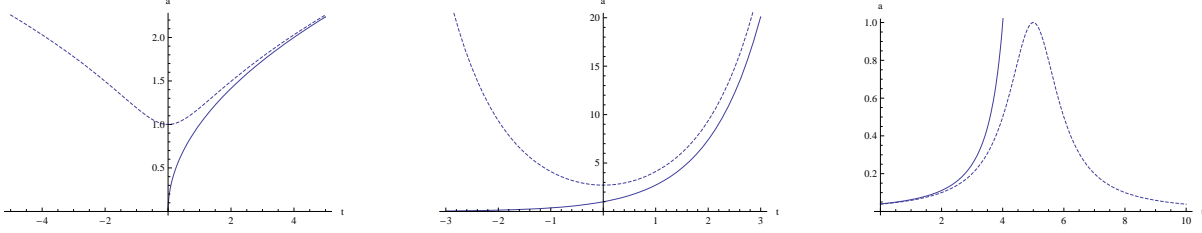


Figure 4: Qualitative behavior of the classical scale factor (solid lines) and its Bohmian or quantum expectation value counterpart (dashed lines). The figures are plotted (from left to right) for $\omega = 1/3$, $\omega = -1$ and $\omega = -4/3$. Note that for the models that exhibit the classical big-bang singularity, (e.g. the left figure), while there is no classical behavior for the negative values of t , in the quantum counterpart of the scale factor t can vary from $-\infty$ to $+\infty$, see (43) and (53).

Again, for the case where the perfect fluid plays the role of a cosmological constant, i.e. $\omega = -1$, using the wave function (40) the above steps lead us the following Bohmian value for the scale factor

$$a(t) \sim e^{\sqrt{\gamma^2 + t^2}}. \quad (56)$$

In comparison with the classical solutions, it is seen that this model represents a bouncing cosmology which recovers the late time behavior of the classical de Sitter Universe (26). In this case the quantum potential takes the form $\mathcal{Q} \sim \frac{1}{\ln^2 a}$, which yields a repulsive force $F_a \sim \frac{1}{a \ln^3 a}$. For $a \rightarrow 0$ (in de Sitter model this occurs in the limit $t \rightarrow -\infty$) F_a takes a large magnitude and thus like the above discussion this force may interpreted as being responsible of the avoidance of this kind of singularity.

To verify how this formalism work for the case of the phantom fluid, we use the duality transformation (13). In this case it is easy to show that the phantom counterpart of the solution (53) reads

$$a_{ph}(t) \sim \left[\gamma^2 + (t_{rip} - t)^2 \right]^{-\frac{1}{3|\omega+1|}}, \quad (57)$$

in which for comparison with the classical solution, we absorb the integration constants in t_{rip} . We see that although this solution has the same limiting behavior as the classical phantom scale factor (12), its behavior near t_{rip} is different. Indeed, while in the classical solution (12), which is valid only in the range $t < t_{rip}$, the scale factor increase monotonically towards the big-rip singularity, its quantum version (57) has a regular behavior at $t = t_{rip}$. This means that the scale factor (57) is free of big-rip singularity at $t = t_{rip}$ and at this moment the expansion behavior of the Universe will be replaced by a contraction phase. The avoidance of the big-rip singularity may be interpreted again by the existence of the quantum potential. Using the duality (13) the quantum potential for the phantom model can be written as

$$\mathcal{Q}_{ph} \sim a^{3|\omega+1|}. \quad (58)$$

Now, we see that the quantum effects show their important role for the large values of the scale factor. Therefore, the repulsive force resulting from this quantum potential (the phantom counterpart of (55)) will take its large magnitude for large values of a and thus the evolution of the scale factor towards the big-rip singularity will be avoided¹.

In figure 4 we have summarized the results of this subsection. As is clear from this figure, for the standard cosmological fluids which we have chosen the cases $\omega = 1/3$ (radiation) and $\omega = -1$ (cosmological constant) here, in conventional inflationary cosmology, the Universe begins with a singularity and expands forever. On the other hand, the evolution of the scale factor based on the quantum-mechanical considerations shows a bouncing behavior in which the Universe bounces from a contraction epoch to a reexpansion era. We see that in the late time of cosmic evolution in which

¹Here, we have taken only a quick look at the quantum phantom model from a phenomenological point of view. In a more fundamental study one may replace the phantom by a scalar field with negative kinetic energy. In this sense the quantum phantom cosmology may be investigated in the framework of scalar field models, see[18].

the quantum effects are negligible these two behaviors coincide with each other. This means that the quantum structure which we have constructed based on the gauge-fixed Lagrangian (23) has a good correlation with its classical counterpart. Also, for the phantom case $\omega = -4/3$, the singular behavior of the classical scale factor at $t = t_{rip}$ is replaced by a transition from an expansion epoch to a recontraction regime. With a glance at this figure one can find the duality between the standard matter solutions (the figure on the left) and the phantom solutions (the figure on the right). This duality may be expressed in terms of the following statements:

- Standard types of matter: There is no classical behavior before the big-bang singularity. The quantum effects dominate in the region of the classical big-bang singularity, i.e., at the small values of scale factor. At the big-bang the quantum solutions bounce from a contraction phase to an expansion era.
- Phantom solutions: There is no classical behavior after the big-rip singularity. The quantum effects dominate in the region of the classical big-rip singularity, i.e., at the large values of scale factor. At the big-rip the quantum solutions fall from an expansion phase to a contraction era.

As a final remark, we would like to emphasize that in comparison with the quantum cosmological model presented in the previous section, one of the advantages of using this formalism in our quantum cosmological model is that, in a natural way, it can offer a time parameter in terms of which we can evaluate dynamical behavior of the cosmic scale factor.

4 Conclusions

In this paper we have studied the classical and quantum dynamics of a perfect fluid cosmological model with an eye to the problem of time gauge in quantum cosmology. To do this, in addition of the Einstein-Hilbert action, we have examined a gauge-fixed Lagrangian which turns out to correspond to a specific choice of time parameter. We have seen that the classical evolution of the Universe based on these two methods has the same solutions represent a late time power law expansion coming from a big-bang singularity when $-1 < \omega < 1$ (except in the case $\omega = -1$ for which we had a de Sitter Universe) and tending to a big-rip singularity when $\omega < -1$. In this sense the selected time parameter in the gauge-fixed method is nothing but the traditional cosmic time in the Einstein-Hilbert Lagrangian. We then dealt with the quantization of the model in which we saw that the classical singular behavior will be modified. In the consensus quantum model, we showed that the Wheeler-DeWitt wave function has a good semiclassical behavior and for large scale factor recovers the classical solutions. Also, for small values of scale factor this wave function shows a pattern in which there are several possible quantum states from which our present Universe could have evolved and tunneled in the past from one state to another. On the other hand, using the gauge-fixed representation at the Lagrangian level, we showed that it may lead to a Schrödinger equation for the quantum-mechanical description of the model. We showed that the Schrödinger equation can be separated and its eigenfunctions can be obtained in terms of analytical functions. By an appropriate superposition of the eigenfunctions, we constructed the corresponding wave packet. The time evolution of this wave packet represents its motion along the larger a direction. As time passes, our results indicated that the wave packets disperse and the minimum width being attained at $t = 0$, which means that the quantum effects are important for small enough t , corresponding to small a . The avoidance of classical singularities due to quantum effects, and the recovery of the classical dynamics of the Universe are another important features of our quantum presentation of the model. These questions have been investigated by two different methods. The time evolution of the expectation value of the dynamical variables and also their Bohmian counterparts have been evaluated in the spirit of the many-worlds and ontological interpretation of quantum cosmology respectively. We verified that a bouncing singularity-free Universe is obtained in both cases. The use of the ontological interpretation has allowed us to understand the origin of the avoidance of singularity by a repulsive force due to the existence of the quantum potential. The repulsive nature of this force prevents the Universe to reach the singularity. We have also taken a glance at the quantum phantom model in this framework

and showed that, in this case, the quantum effects dominate in the region of the classical big-rip singularity, i.e., the quantum effects occur at large values of the scale factor. We saw that the quantum-mechanical considerations in the phantom model result in a scale factor which has a regular behavior at the big-rip moment, changing its evolution from an expansion era to a contraction epoch. Therefore, the big-rip singularity may be removed in the quantum theory.

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